Quandle homology and

the fundamental 3-classes of knot group representations

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Announce; my book on quandle theory

Title;

Quandle theory— Relative topology, knots, and cohomology

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Published; in Springer briefs

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Basics on quandles, colorings, qn'dl homology

& a summary of my results, and my courses to study qn'dl.

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A simple question on quandle

For what top. objects is quandle useful?

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My answers

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My answers .

- **We can process the pair $H \subset G$ as one object.**

e.g., Waldhausen thm. w/ $G := \pi_1(S^3 \setminus K) \supset \pi_1(\partial S^3 \setminus K) =: H$.

- **Reduction of dim. and non-compactness of G,**

e.g., cases of $\dim(H \setminus G) \leq \dim G$, co-compact subgrp H.

Today.

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e.g., cases of $\dim(H \setminus G) \leq \dim G$, co-compact subgrp H.

Today. **I apply this philosophy to the fundamental 3-class, Q'dl theory shall give a 2-dim. computation of the inv. as in** $G = \pi_1(S^3 \setminus K) \supset \pi_1(\partial S^3 \setminus K)$ **with** $\dim(H \setminus G) = 2$**,**

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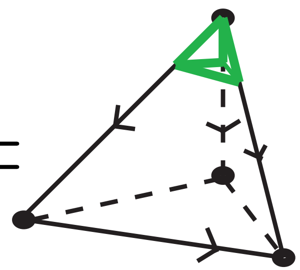
$E := S^3 \setminus \nu L$: link complement with triangulation

Theme. The fund. 3-class $[E,\ \partial E] \in H_3(E,\ \partial E; \mathbb{Z}) \cong \mathbb{Z}$.

Here, $[E,\ \partial E] := \sum_{\text{3-simplexes}} \pm$ 

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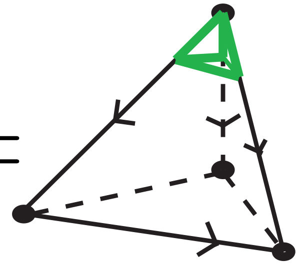
Here, $[E,\ \partial E] := \sum_{\text{3-simplexes}} \pm$ 

Def. Relative fund. 3-class w.r.t. $H \subset G$ grps.

INPUT $f : \pi_1(S^3 \setminus L) \longrightarrow G$ s.t. $f(\partial(S^3 \setminus \nu L)) \subset H$.

$\phi : G^3 \longrightarrow A$ "relative" group 3-cocycle.

OUTPUT: $\langle\ f^*(\phi),\ [E_L,\ \partial E_L]\ \rangle \in A$

POINT: The rel. 3-class seems uncomputable from def.

(Again) <u>Relative fund. 3-class</u> $H \subset G$ grps.

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Examples

- **Hyperbolic volume** [W. Neumann, Zickert],

  if f: the holonomy, $G = SL_2(\mathbb{C})$, ϕ: Chern-Simons 3-class.

- It is called **Dijkgraaf-Witten invariants**, if $|G| < \infty$.

- **Milnor link inv.**, if the settings are "nilpotent" [Turaev, Porter]

- **Triple cup products**, if ϕ is a trilinear map (seen later)

$\Longrightarrow$ So,

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$\Longrightarrow$ So, I hoped theorems in such a general setting.

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L: non-cable (prime) knot, or hyperbolic link.
$\Longrightarrow$
the output is computable from only link diagrams without describing triangulation of $S^3 \setminus L$.

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Theorem 2 (N.)

If L is a knot and $H \subset G$ is "malnormal",

$\Longrightarrow$

the computation forms a "quandle cocycle inv."

- $H \subset G$ is malnormal, if $gHg^{-1} \cap H = 1$ for $^\forall g \in G - H$.

Contents & Plan

§1 Oultlines of this talk, and the proof.

§2 Quandle homology and Hochshild relative homology.

§3 Inoue-Kabaya chain map & Some results.

§4 Main result, and an outline of the proof.

§5 An example; Triple cup products.

Strategy of this talk

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(Rough) Shadow 3-cocycle inv. [Carter-Saito-Kamada]

INPUT $\quad X$-coloring $\mathcal{C} :\ \{\text{ arcs of } D\ \} \longrightarrow X = G/H$.

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OUTPUT: $\quad \langle\, \mathcal{C}^*(\phi),\ [Q_L]\,\rangle \in A$

Here $[Q_L]$ is a q'dl 3-cycle, or a weight sum from a diagram.

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$\Uparrow$ $^{\exists}$?Reduction: KEY "Inoue-Kabaya map".

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Rough explanation

What is the IK map?

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$$\boldsymbol{\varphi_* : C_*^Q(X) \longrightarrow C_*^N(X)_G}$$

"**Qn'dl complex**" "**Hochshild rel. complex**" $\leftarrow$ **§2**

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"Qn'dl complex"   "Hochshild rel. complex" ← §2

Section 3 introduces Inoue-Kabaya map.

I also explain why malnormality is compatible with for φ_*.
Before them, I now fix settings of quandles:

Quandles in this talk Assume Malnormality of $H \subset G$.

Def Suppose $z_0 \in G$ s.t. ${}^\forall h \in H \quad hz_0 = z_0 h$.

$X := H \backslash G$.

$[g] \lhd [h] := [z_0^{-1} g h^{-1} z_0 h]$.

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Ex1 The parabolic quandle $X = G/H$

when $H := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}_{a,b \in \mathbb{C}} \subset G := PSL_2(\mathbb{C})$.

$\uparrow$ **Fact.** malnormality.

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$\uparrow$ **Fact.** malnormality.

Ex2 The knot quandle ($X = G/H$) of $K \subset S^3$

when $G := \pi_1(S^3 \setminus K)$ knot grp. & $H := \pi_1(\partial(S^3 \setminus K))$

Thm. (J. Simon '76, Harpe and C. Weber' 11)

The pair $H \subset G$ is malnormal $\iff$

K is neither a composite knot nor a cable knot.

Contents & Plan

Review; Quandle complex

$C_n^R(X) := \mathbb{Z}\langle (x_1, \ldots, x_n) \in X^n \rangle$.

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$$\partial_2(a, b) \quad := (a) - (a \lhd b)$$

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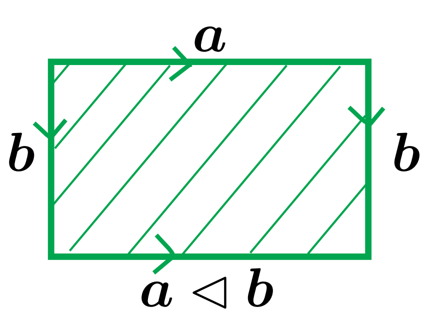

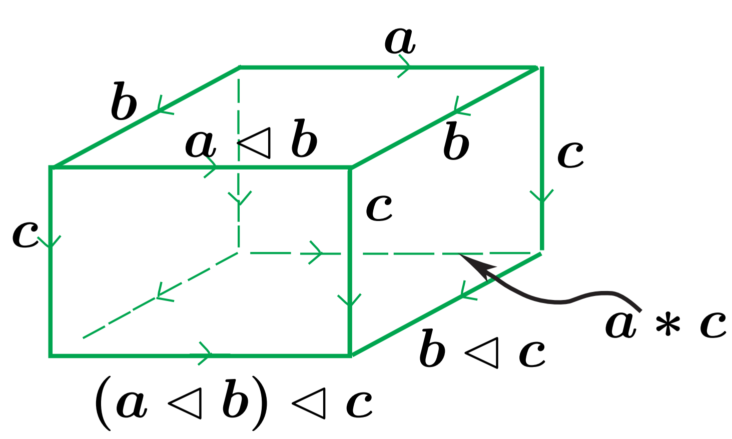

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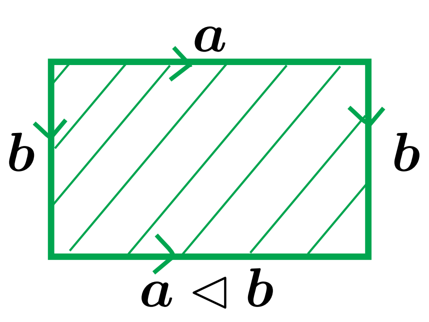
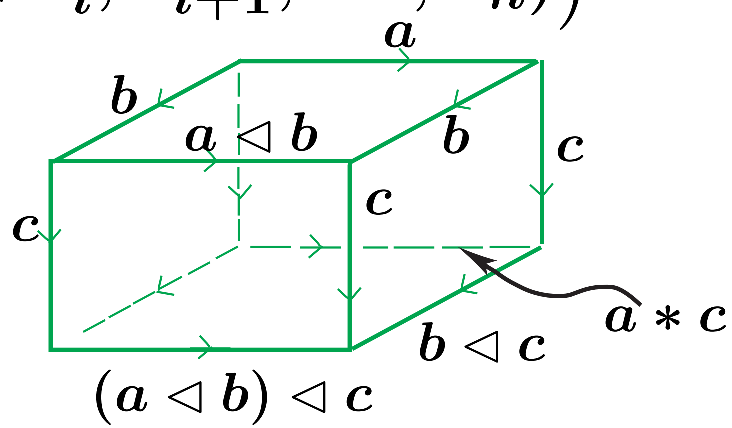
$$\partial_n(x_1, \dots, x_n) := \sum_{i=1}^{n} (-1)^i \Big((x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ -(x_1 \lhd x_i, \dots, x_{i-1} \lhd x_i, x_{i+1}, \dots, x_n) \Big)$$





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$C_n^D(X) := \mathbb{Z}\langle (x_1, \ldots, x_n) \in X^n \mid {}^{\exists}i,\ x_i = x_{i+1} \rangle$.

$C_n^Q(X) := C_n^R(X)/C_n^D(X)$.

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Methods to compute H_n^Q.

- by computer programmings, when $|X|$ is small.
- Eisermann gave a computation of $H_2^Q(X; \mathbb{Z})$.
- T.Mochizuki computed $H_3^Q(X; \mathbb{F}_q)$ for Alex. qn'dl.
- I gave a computation of $H_3^Q(X; \mathbb{Z})$ for some qn'dl.

Hochshild relative grp. complex where $X = H \backslash G$.

$$C_n^{\Delta}(X) := \mathbb{Z}\Big\langle (x_0, \ldots, x_n) \in X^{n+1} \quad \Big\rangle \qquad (\leftarrow (n+1)\text{-tuples}).$$

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$C_n^{\Delta}(X)_G := C_n^{\Delta}(X)$/diagonal action of G.

$$\partial_n(x_0, x_1, \ldots, x_n) := \sum_{i=0}^{n} (-1)^i (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Rem.

Hochshild relative grp. complex where $X = H\backslash G$.

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Rem. In general, $H_n^{\Delta}(X)_G$ is mysterious.

But, in malnormal case, the complex subject to the degenerated part becomes some familiar complexes, as follows;

Normalized Hochshild complex where $X = H \backslash G$.

$$C_n^{\mathrm{D}}(X) := \mathbb{Z}\Big\langle (x_0, \ldots, x_n) \;\Big|\; {}^{\exists} i,\ x_i = x_{i+1} \Big\rangle \subset C_n^{\Delta}(X).$$

$$C_n^{N}(X)_G := \Big(C_n^{\Delta}(X) / C_n^{\mathrm{D}}(X)\Big) / \text{the diag. action of } G.$$

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Lem. If $H \subset G$ is malnormal & $|G/H| \geq \infty$,

$C_*^N(X)$ is a "semi"-free $\mathbb{Z}[G]$-module, and acyclic.

In particular, $H_n^N(X)_G \cong H_n^{\mathrm{group}}(G, H; \mathbb{Z})$.

$$\cdots \to H_n^{\mathrm{gr}}(H) \to H_n^{\mathrm{gr}}(G) \to H_n^{\mathrm{gr}}(G, H) \to H_{n-1}^{\mathrm{gr}}(H) \to \cdots$$

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p.f. Show the action $C_*^N(X) \curvearrowleft G$ is free, i.e., Stab $= 1_G$.
Suppose $x_i \cdot g = x_i$ and $x_{i+1} \cdot g = x_{i+1} \in X$, with $x_i \neq x_{i+1}$
i.e. ${}^{\exists}h, k \in H$ s.t. $x_i g = h x_i$ and $x_{i+1} g = k x_{i+1} \in G$,
$\Longrightarrow h = x_{i+1} x_i^{-1} k x_i x_{i+1}^{-1} \in H \cap (x_i x_{i+1}^{-1})^{-1} H x_i x_{i+1}^{-1} = 1$.
Malnormalitiy means $h = 1$. Hence $g = 1$. $\square$

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Def. **Inoue-Kabaya chain map.** Fix $z_0 \in H$.

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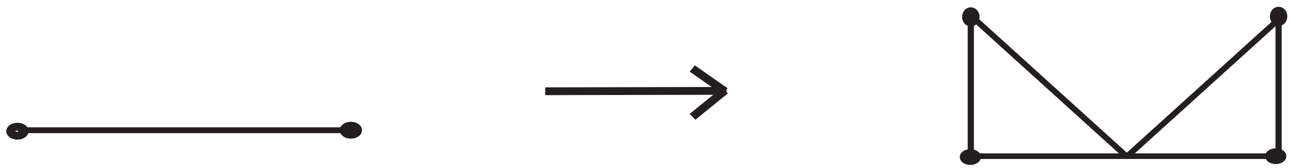
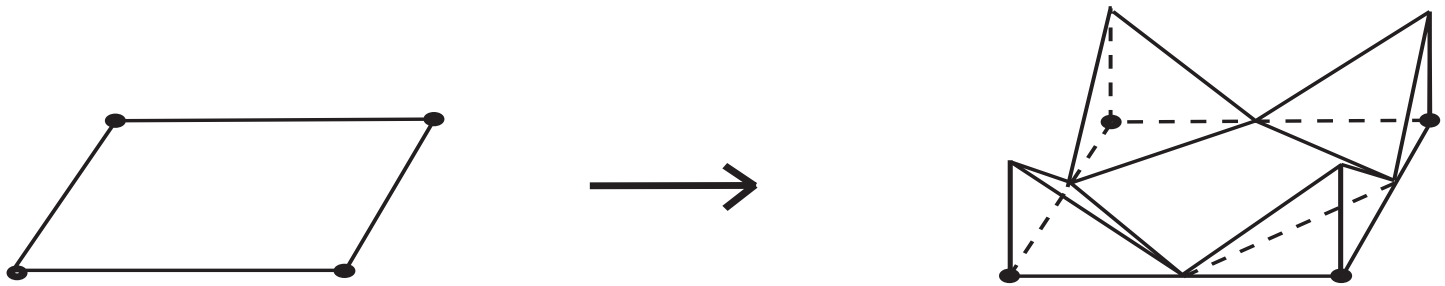


- $n = 3$



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\end{aligned}$$

Thm.[IK]
This map is a chain map. i.e., $\partial_* \circ \varphi_* = \varphi_* \circ \partial_*$

In summary, we get

$$\varphi_* : H_*^Q(X) \longrightarrow H_*^N(X)_G \cong H_*^{\mathrm{gr}}(G, H).$$

Known result 1. Inoue-Kabaya chain map.

Thm. [Inoue-Kabaya]

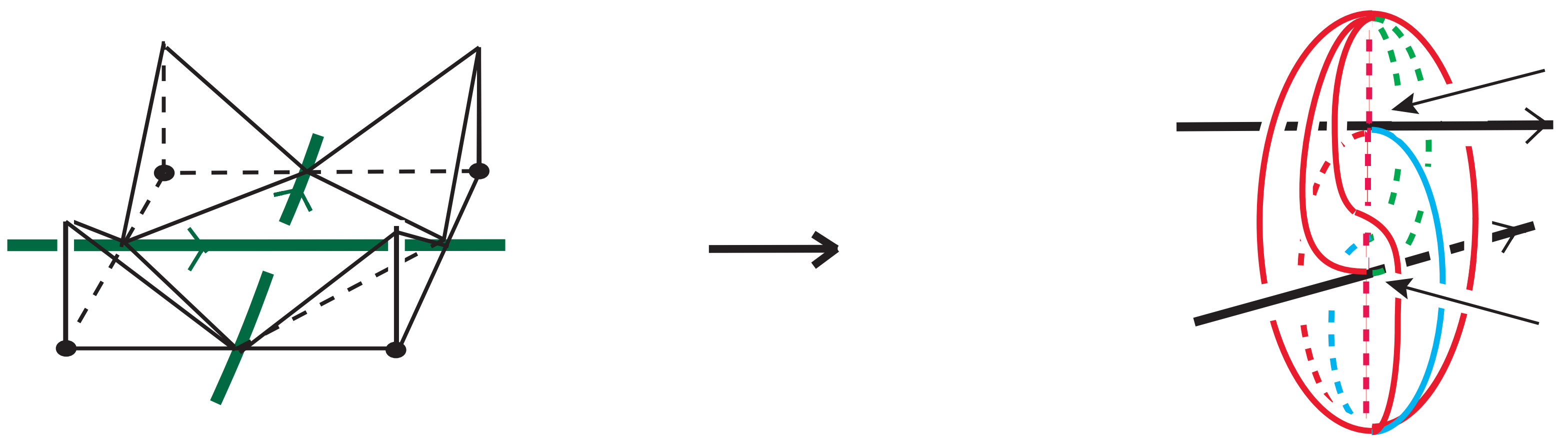
Let $G = PSL_2(\mathbb{C}) \supset H := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}_{a,b\in\mathbb{C}}$ as before,

Let $\theta : H^3_{\text{grp.}}(G, H; \mathbb{C}/\mathbb{Z})$ Chern-Simons 3-class

Regard the holonomy $f : \pi_L \to G$ as an X-coloring,

$\Longrightarrow$

The associated shadow inv. $\langle\, f^*(\theta),\ [Q_L]\,\rangle \in A$ is equal to the (complex) volume of L, if L is hyperbolic.



This thm will be explained in the next talk.

Known result 2. Inoue-Kabaya chain map.

Thm.[N.14]

If $|X| < \infty$ and X is "of type k",
G is "a univ. central extended grp., up to k-torsions",
then $\varphi_* : H_3^Q(X) \cong H_3^Q(G)$, (up to k-torsions)

Thm.[N.14]

Let $X = \mathbb{F}_q$ be "an Alexander quandle on $\mathbb{F}_q$",
Then, every quandle 3-cocycle of [T. Mochizuki] is recovered from some group 3-cocycles by the IK map.

Thm.[N.13]

If X is a G-family of Alexander quandles, I got methods to produce q'dl cocycles from invariant theory.

Contents & Plan

Today result. Inoue-Kabaya chain map.

Thm.[N]

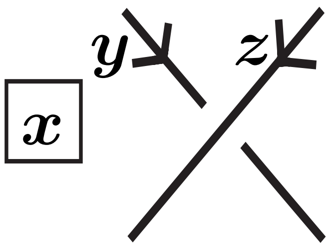
Let K be a non-composite and non-cable knot.

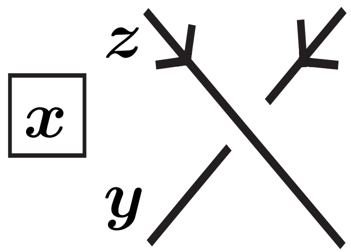
Then, $H_3^Q(Q_K;\mathbb{Z}) \cong \mathbb{Z}$ is generated by $[Q_L]$.

Moreover, the IK-map is an isomorphisms
$\varphi_* : H_3^Q(Q_K) \to H_3^N(Q_K)_G \cong H_3(E_K, \partial E_K) \cong \mathbb{Z}$.

Here $[Q_L] \in C_3^Q(Q_K;\mathbb{Z})$ is the formal sum

$$\stackrel{\text{def}}{=\!=} \sum_{\tau:\text{crossing of } D} \pm (x, y, z).$$





Today result. Inoue-Kabaya chain map.

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P.f. Step 1. I show $H_3^Q(Q_K) \cong \mathbb{Z}$ by homotopy th'ry.

$\Longrightarrow \quad {}^{\exists}k \in \mathbb{Z}, \quad \varphi_*(Q_K) = k \cdot [E_K, \partial E_K]$.

Claim $k = \pm 1$.

(Step 2) For hyperbolic knots, $k = \pm 1$ by IK's result.

(Last step) By the JSJ decomposition, and the excision axiom $\square$.

Proof of Thm. 1 w.r.t. $f : \pi_L \to G$.

Quandle cpx. | **Hochshild cpx.** | **Rel. grp. cpx.**

$$
\begin{array}{ccccc}
H_3^Q(Q_L) & \xrightarrow[\cong]{\varphi_*} & H_3^N(Q_L)_{\pi_L} & \xleftarrow[\cong]{\alpha} & H_3^{\mathrm{gr}}(\pi_1 E_L, \pi_1(\partial)) \\
\downarrow f_* & & \downarrow f_* & \overset{\beta}{\curvearrowright} & \downarrow f_* \\
H_3^Q(X) & \xrightarrow{\varphi_{\mathrm{IK}}} & H_3^N(X)_G & \xleftarrow{\alpha} & H_3^{\mathrm{gr}}(G, H) \\
 & & & & \downarrow \langle \phi, \bullet \rangle \\
 & & & & A
\end{array}
$$

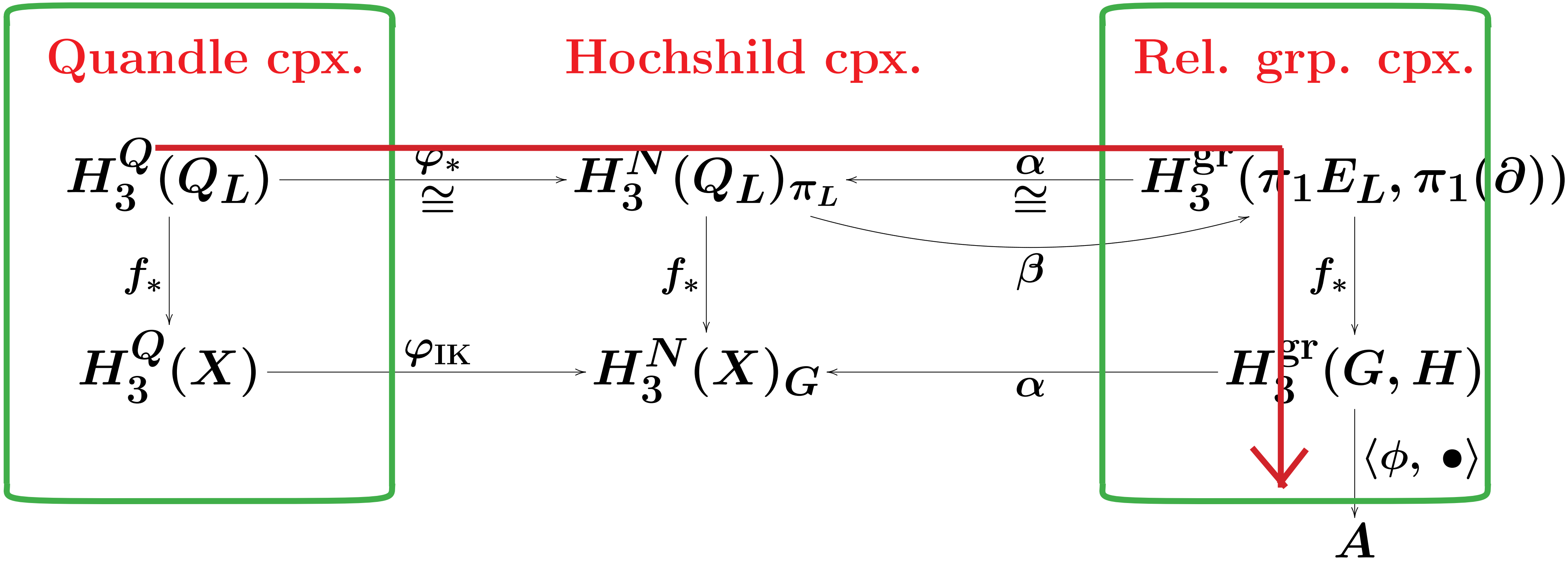
Proof of Thm. 1 w.r.t. $f : \pi_L \to G$.

Quandle cpx. | **Hochshild cpx.** | **Rel. grp. cpx.**

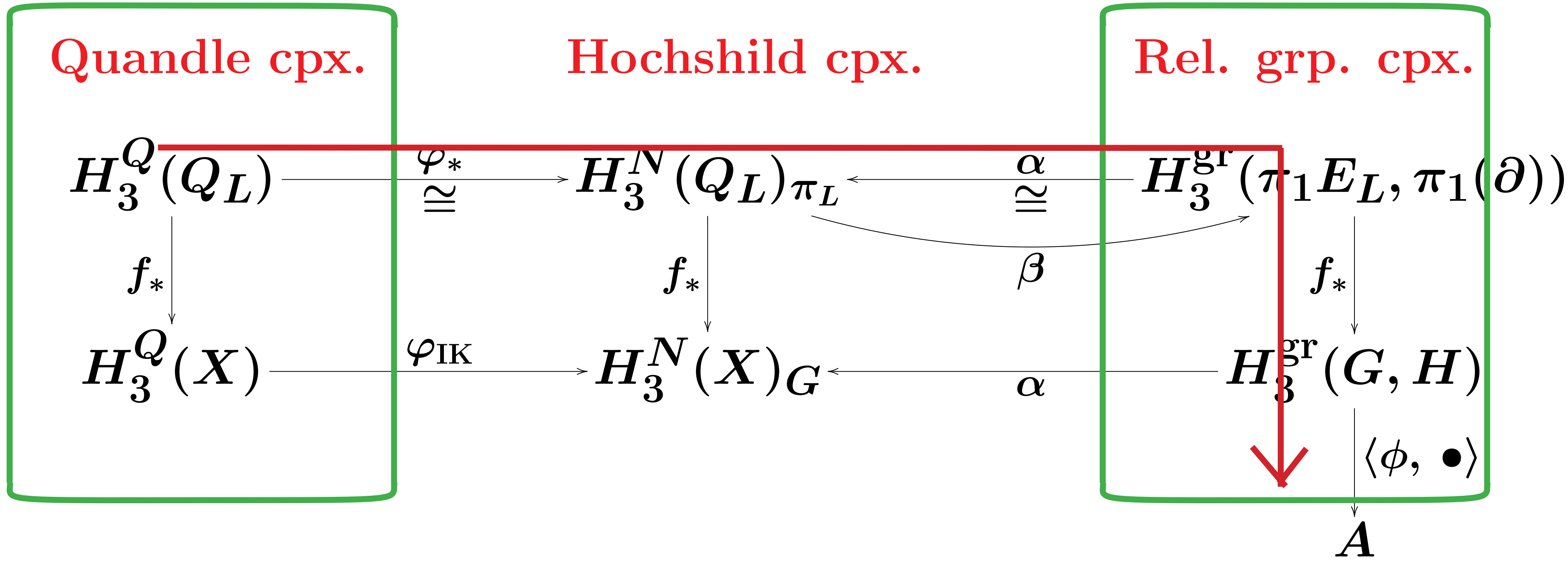
$$
\begin{array}{ccccc}
H_3^Q(Q_L) & \xrightarrow[\cong]{\varphi_*} & H_3^N(Q_L)_{\pi_L} & \xleftarrow[\cong]{\alpha} & H_3^{\mathrm{gr}}(\pi_1 E_L, \pi_1(\partial)) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
H_3^Q(X) & \xrightarrow{\varphi_{\mathrm{IK}}} & H_3^N(X)_G & \xleftarrow{\alpha} & H_3^{\mathrm{gr}}(G, H) \\
 & & & & \downarrow \langle \phi, \bullet \rangle \\
 & & & & A
\end{array}
$$

$$\beta : H_3^N(Q_L)_{\pi_L} \to H_3^{\mathrm{gr}}(\pi_1 E_L, \pi_1(\partial))$$

Proof of Thm. 1 w.r.t. $f : \pi_L \to G$.



Proof of Thm. 1 w.r.t. $f : \pi_L \to G$.



Theorem 1 (N.)

L: non-cable (prime) knot, or hyperbolic link.
$\Longrightarrow$
the output is computable from only link diagrams

p.f. $\langle\, f^*(\phi),\ [E_L,\ \partial E_L]\,\rangle = \langle\, \varphi^* \circ \beta^* \circ f^*(\phi),\ [Q_L]\,\rangle.$

The RHS is diag. computable. □

Special case, Triple cup product:

(Invariants of hom $f : \pi_1(E) \longrightarrow G$, where $E := S^3 \setminus \nu L$)

Input M : right G-module/ a ring A

$$\phi : M^3 \xrightarrow{\text{trilinear}} A \quad \text{s.t. } \phi(a\cdot g, b\cdot g, c\cdot g) = \phi(a,b,c).$$

Output: $$H^1(E, \partial E; M)^3 \xrightarrow{\text{cup prod.}} H^3(E, \partial E; M^{\otimes 3}) \to$$

$$\xrightarrow{\bullet \cap \text{rel. fund. 3-class}} M \otimes M \otimes M \xrightarrow{\langle \phi, \bullet \rangle} A.$$

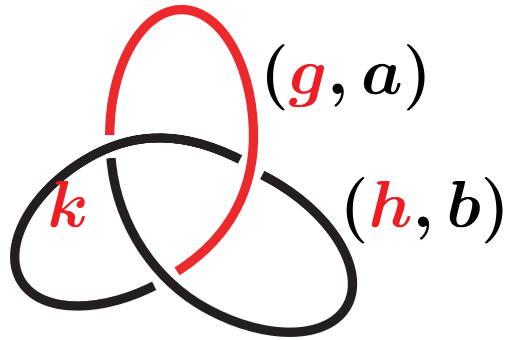
Theorem 3 (N.)

If L is a prime knot or hyperbolic link.

$\Longrightarrow$

the trilinear output is "diagrammatically" computable.

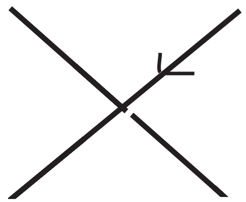
On $H^1(E, \partial E; M)$ w.r.t. $\pi_1(S^3 \setminus L) \xrightarrow{f} G$



$\{$ over-arcs $\} \xrightarrow{\text{map}} G$: Wirtinger pre. of f.

Def.[IIJO]. (M: right $\mathbb{Z}[G]$-module)

- Coloring over f is $\mathcal{C} : \{$ over-arcs $\} \to M \times G$ over f

  s.t. (x, g)  $(y, h) \in M \times G$

  $\Downarrow$

  $(y + (x - y) \cdot h,\ h^{-1}gh) \in M \times G$

- A shadow of $\mathcal{C}$ is $\lambda : \{$ regions of D $\} \to M$ s.t.

  (i) (b, h)  a | $b + (a - b) \cdot h$ (ii) $\lambda(\{\infty\}) = 0 \in M$.

Thm.[N.14]

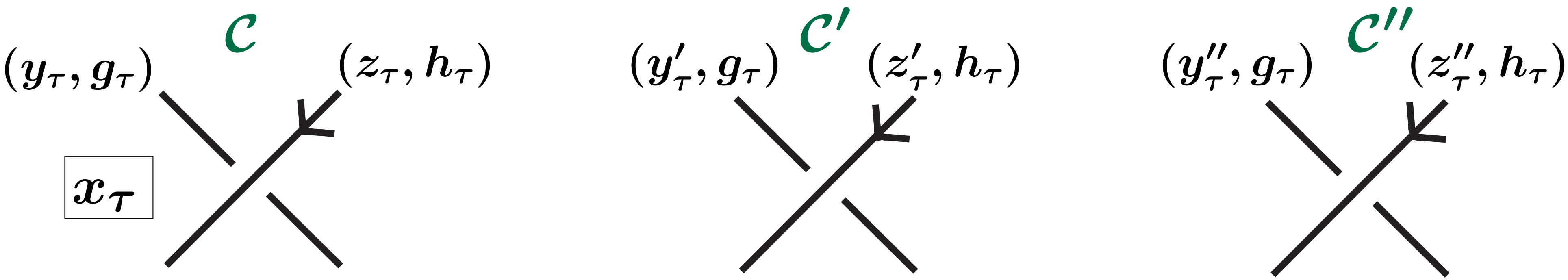
$^\exists$isom. : $\mathrm{SCol}(D_f) \cong H^1(E, \partial E; M) \oplus M$.

Def.[N.] $\mathcal{T}_\phi : (\mathrm{SCol}(D_f))^3 \to A$.

For shadow colorinsg $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ $\in \mathrm{SCol}(D_f)$, we define

$$\mathcal{T}(\mathcal{C}, \mathcal{C}', \mathcal{C}'') := \sum_{\tau:\ \text{crossing}} \phi\left((x_\tau - y_\tau)(1 - g_\tau^{\epsilon_\tau}),\ y''_\tau - z''_\tau,\ z''_\tau\,(1 - h_\tau^{-1})\right).$$

Here, the three colorings around the τ are given by



Prop.(Invariance w.r.t. f)[N.]

If $D \overset{R\text{-moves}}{\longleftrightarrow} D'$,

$$\mathrm{SCol}(D_f)^3 \overset{\exists\text{isom.}}{\longleftrightarrow} \mathrm{SCol}(D'_f)^3, \quad \mathcal{T}_\phi : \mathrm{SCol}(D_f)^3 \to A, \quad \mathcal{T}'_\phi : \mathrm{SCol}(D'_f)^3 \to A \quad (\circlearrowleft)$$

Thm.[N.]

$^{\exists}$**Isom** : $\mathrm{SCol}(D_f) \cong H^1(E, \partial E; M) \oplus M$.

Further, if L is a non-cable (prime) knot or a hyperbolic link,

$\mathcal{T}_\phi : \mathrm{SCol}(D_f)^3 \to A$ agrees with the triple product.

(Again) (hom $f : \pi_1(E) \longrightarrow G$, where $E := S^3 \setminus \nu L$)

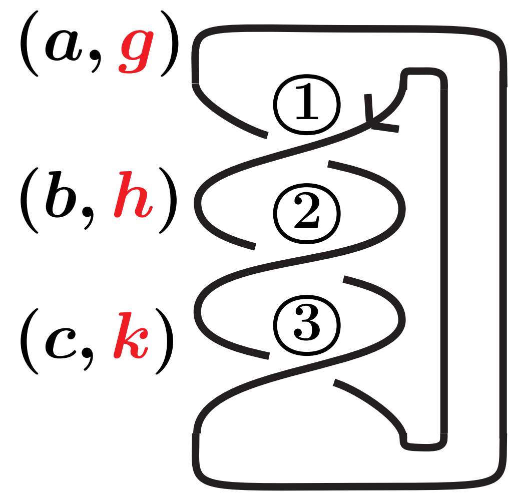
Input M : right G-module/ a ring A

$$\phi : M^3 \xrightarrow{\text{trilinear}} A \quad \text{s.t.} \quad \phi(a\cdot g, b\cdot g, c\cdot g) = \phi(a, b, c).$$

Output: $H^1(E, \partial E; M)^3 \xrightarrow{\text{cup prod.}} H^3(E, \partial E; M^{\otimes 3}) \to$

$$\xrightarrow{\bullet \cap \text{rel. fund. 3-class}} M \otimes M \otimes M \xrightarrow{\langle \phi,\ \bullet \rangle} A.$$

Exa. (Trefoil knot) $\langle\ g, h \mid ghg = hgh\ \rangle$,



Col. cond.

$$\begin{cases} c = a \cdot h + b \cdot (1 - h) \\ a = b \cdot k + c \cdot (1 - k) \\ b = c \cdot h + a \cdot (1 - g) \end{cases}$$

$$\Longleftrightarrow \quad (a-b)\cdot(1-g+hg) = (a-b)\cdot(1-h+gh) = 0. \qquad (\clubsuit)$$

$$\mathcal{T}_\phi\Big((a,b),(a',b'),(a'',b'')\Big) \overset{\text{by def}}{=\!=}$$

$$+\phi\Big(-a\cdot(1-g),\ a'-b',\ b''\cdot(1-h^{-1})\Big) \quad \cdots ①$$

$$+\ \phi\Big(-b\cdot(1-h),\ b'-c',\ c''\cdot(1-k^{-1})\Big) \quad \cdots ②$$

$$+\ \phi\Big(-c\cdot(1-k),\ c'-a',\ a'\cdot(1-g^{-1})\Big) \quad \cdots ③$$

$$= \cdots\cdots = \phi\Big((a-b)g^{-1},\ (a'-b')\cdot h,\ (a''-b'')\Big) \in A.$$

Summary

- $\dim(H \backslash G) \leq \dim G$, when $G = \pi_1(S^3 \setminus K)$

Theorem 1 (N.)

L: non-cable (prime) knot, or hyperbolic link.
$\Longrightarrow$
the output is computable from only link diagrams.

Theorem 2 (N.)

If L is a knot and $H \subset G$ is "malnormal",
$\Longrightarrow$
the computation forms a "quandle cocycle inv."

HOPE: Continued study of fundamental 3-classes of knots

Thank you